Virtual shadow modules and their link invariants

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Abstract

We introduce an algebra $\mathbb{Z}[X, S]$ associated to a pair X, S of a virtual birack X and X-shadow S. We use modules over $\mathbb{Z}[X, S]$ to define enhancements of the virtual birack shadow counting invariant, extending the birack shadow module invariants to virtual case. We repeat this construction for the twisted virtual case. As applications, we show that the new invariants can detect orientation reversal and are not determined by the knot group, the Arrow polynomial and the Miyazawa polynomial, and that the twisted version is not determined by the twisted Jones polynomial.

KEYWORDS: Biracks, birack shadows, virtual links, twisted virtual links, link invariants, enhancements of counting invariants

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1 Introduction

Virtual knots and links were introduced in [18] and have been the subject of much study since. Twisted virtual knots and links were introduced in [4] and have been studied in papers such as [9, 15, 17]. Biracks (including biquandles) were first introduced in [13] as an algebraic structure defining invariants of framed knots and links in S^3 . Biquandle-based representational invariants of virtual knots and links were studied in papers such as [7, 12, 23]. Biquandles were generalized to virtual biquandles, structures involving operations at virtual crossings as well as classical crossings, in [19]. In [21] a representational invariant of unframed classical and virtual knots and links, the integral birack counting invariant $\Phi_X^{\mathbb{Z}}(L)$, was defined. Enhancements of $\Phi_X^{\mathbb{Z}}(L)$, i.e., invariants which specialize to $\Phi_X^{\mathbb{Z}}(L)$ but are generally stronger invariants, have been studied in various papers such as [3, 8].

In [1] an associative algebra known as the rack algebra was defined from a finite rack X. In [5] quandle algebras were used to enhance the quandle counting invariant, and later in [14] a modified rack algebra was used to enhance the rack counting invariant. In [3], rack algebras were generalized to birack algebras. In [22], birack algebras were further generalized to birack shadow algebras associated to pairs X, S where X is a birack and S is an X-shadow, i.e. a set with an X-action satisfying certain diagrammatically motivated properties. Birack shadow algebra invariants as defined in [22] are well defined for classical knots and links but not for virtual knots and links.

In this paper we introduce *virtual birack shadow algebras* and *twisted virtual birack shadow algebras*. As an application, we use modules over these algebras to enhance the virtual birack and twisted virtual birack counting invariants, and we show that the enhanced invariants can detect orientation reversal and are not determined by the knot group, the Arrow or Miyazawa polynomials, or the twisted Jones polynomial.

The paper is organized as follows. In section 2 we recall the basics of virtual knots, virtual biracks and virtual birack shadows, then introduce virtual birack shadow algebras and modules and use these to define

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an infinite family of enhanced virtual link invariants. In section 3 we introduce twisted virtual birack shadow algebras and modules and use these to define an infinite family of enhanced twisted virtual link invariants. We collect a few computations and examples in section 4, and we conclude in section 5 with a few open questions for further research.

2 Virtual birack shadow algebras

2.1 Virtual knots and links

Virtual knots and links were introduced in [18] by Louis Kauffman in 1996 as a combinatorial generalization of classical knots and links. Every oriented knot or link diagram is a planar 4-valent directed graph with crossing information at the vertices. Edges in this graph are called *semiarcs*. Such a graph can be encoded as a *signed Gauss code* by naming the crossings, choosing a base point on each component, and then noting the order in which the over and under instances of each crossing are encountered when following the orientation of each component.



A virtual link is then an equivalence class of signed Gauss codes under the Gauss code versions of the Reidemeister moves. Attempting to reconstruct the original oriented link diagram from a signed Gauss code, one quickly finds that some signed Gauss codes determine nonplanar diagrams, i.e. diagrams needing extra crossings not listed in the Gauss code. An equivalence class of signed Gauss codes is a *classical link* if it contains a signed Gauss code corresponding to a planar oriented link diagram.

For the nonplanar diagrams, Kauffman introduced *virtual crossings* drawn as circled self-intersections which do not appear in Gauss codes along with the rule that two virtual link diagrams are equivalent if their signed Gauss codes are equivalent by Gauss code Reidemeister moves. In terms of virtual knot diagrams, this breaks down into the *virtual Reidemeister moves*:



Geometrically, a virtual crossing can be understood as representing genus in the surface on which the link diagram is drawn; thus, virtual knot theory is closely related to the theory of knots and links in *I*-bundles

over compact surfaces [16].



Alternative interpretations of virtual knot diagrams exist, such as Dror Bar-Natan's circuit board analogy in which classical crossings represent logic gates on a circuit board and virtual crossings represent the connections between the gates (where we don't care which wire goes over or under).

Including virtual crossings restores the planarity of our knot diagram. The edges in a virtual knot diagram considered as a planar 4-valent graph with crossing information (classical or virtual) specified at each vertex will still be called *semiarcs*; we also define a *classical semiarc* to be the result of dividing our virtual knot diagram only at classical crossing points, i.e. edges in the original nonplanar graph.

2.2 Virtual Biracks

A virtual birack is an algebraic structure whose axioms encode the blackboard framed virtual isotopy moves, obtained from the virtual isotopy moves by replacing the usual Reidemeister type I move with the blackboard framed Reidemeister type I move:



Definition 1 Let X be a set and define $\Delta : X \to X \times X$ by $\Delta(x) = (x, x)$. A virtual birack structure on X is a pair of invertible maps $B, V : X \times X \to X \times X$ satisfying the conditions

(i) B and V are sideways invertible, that is there exist unique invertible maps $S, vS : X \times X \to X \times X$ satisfying for all $x, y \in X$

$$S(B_1(x,y),x) = (B_2(x,y),y)$$
 and $vS(V_1(x,y),x) = (V_2(x,y),y)$

- (ii) *B* is *diagonally invertible*, that is, the components $(S^{\pm 1} \circ \Delta)_1 : X \to X$ and $(S^{\pm 1} \circ \Delta)_2 : X \to X$ of the compositions $S \circ \Delta$ and $S^{-1} \circ \Delta$ are bijections,
- (iii) V is self-inverse and diagonal fixing, that is, $V^2 = \mathrm{Id}_{X \times X}$ and $(vS \circ \Delta)_1 = (vS \circ \Delta)_2$;
- (iv) B, V satisfy the set-theoretic Yang-Baxter equations:

$$(B \times \mathrm{Id}_X)(\mathrm{Id}_X \times B)(B \times \mathrm{Id}_X) = (\mathrm{Id}_X \times B)(B \times \mathrm{Id}_X)(\mathrm{Id}_X \times B),$$

$$(V \times \mathrm{Id}_X)(\mathrm{Id}_X \times V)(V \times \mathrm{Id}_X) = (\mathrm{Id}_X \times V)(V \times \mathrm{Id}_X)(\mathrm{Id}_X \times V), \text{ and }$$

$$(B \times \mathrm{Id}_X)(\mathrm{Id}_X \times V)(V \times \mathrm{Id}_X) = (\mathrm{Id}_X \times V)(V \times \mathrm{Id}_X)(\mathrm{Id}_X \times B).$$

We will find it convenient to abbreviate $B_1(x,y) = y^x$, $B_2(x,y) = x_y$, $V_1(x,y) = y^{\widetilde{x}}$, and $V_2(x,y) = x_{\widetilde{y}}$.

We also have

Definition 2 Let X and Y be sets with virtual birack maps B_X, V_X and B_Y, V_Y . A virtual birack homomorphism is a map $f: X \to Y$ satisfying

$$B_Y \circ (f \times f) = (f \times f) \circ B_X$$
 and $V_Y \circ (f \times f) = (f \times f) \circ V_X$.

Example 1 Examples of virtual birack structures include:

- Racks. A set X with a self-distributive right-invertible right action $\triangleright : X \times X \to X$ is a virtual birack under the maps $B(x, y) = (y \triangleright x, x)$ and V(x, y) = (y, x).
- (v, t, s, r)-Biracks. Let $\check{\Lambda} = \mathbb{Z}[v^{\pm 1}, t^{\pm 1}, s, r^{\pm 1}]/(s^2 (1 tr)s)$. Then any $\check{\Lambda}$ -module X is a virtual birack under the operations B(x, y) = (ty + sx, rx) and $V(x, y) = (vy, v^{-1}x)$.
- Constant action virtual biracks. Let X be any set and let $\sigma, \tau, \nu : X \to X$ be bijections. Then $B(x,y) = (\tau(y), \sigma(x)), V(x,y) = (vy, v^{-1}x)$ defines a virtual birack structure on X iff $\sigma\tau = \tau\sigma$, $\sigma\nu = \nu\sigma$ and $\tau\nu = \nu\tau$.
- Fundamental virtual birack of a framed virtual link. For a blackboard framed virtual link diagram L, let G be a set of generators corresponding bijectively with the semiarcs of L. The set W(L) of virtual birack words in L is defined recursively by the rules

(i)
$$g \in G \Rightarrow g \in W(L)$$
 and

(ii)
$$g, h \in W(L) \Rightarrow B_i^{\pm 1}(g, h), S_i^{\pm 1}(g, h) \in W(L), V_i^{\pm 1}(g, h), \text{ and } vS_i^{\pm 1}(g, h) \in W(L) \text{ where } i = 1, 2.$$

The set of equivalence classes of W(L) under the equivalence relation generated by the crossing relations in definition 4 and the virtual birack axioms is then a virtual birack whose isomorphism class is independent of the diagram chosen to represent L. This virtual birack is called the *fundamental virtual birack* of L, denoted VB(L).

Definition 3 Let $X = \{x_1, \ldots, x_n\}$ be a finite set with virtual birack structure maps B, V. We can conveniently specify the maps B, V with a virtual birack matrix

$$M_X = \left[\begin{array}{c|c} B_1 & B_2 & V_1 & V_2 \end{array} \right]$$

where if $B(x_i, x_j) = (x_k, x_l)$ and $V(x_i, x_j) = (x_m, x_n)$ then $(B_1)_{j,i} = k$, $(B_2)_{i,j} = l$, $(V_1)_{j,i} = m$ and $(V_2)_{i,j} = n$. These matrices can be understood as operation tables for the operations $(x_j)^{(x_i)}$, $(x_i)_{(x_j)}$, $(x_j)^{(\overline{x_i})}$, and $(x_i)_{(\overline{x_j})}$ respectively. Note the reversed order of i, j in B_1 and V_1 ; this convention is chosen so that the rows and outputs represent the same strand while the columns represent the other strand crossing over, under or virtually respectively.

Example 2 Let $X = \mathbb{Z}_5 = \{1, 2, 3, 4, 5\}$. Then the (v, t, s, r)-virtual birack structure on X given by v = 2, t = 3, s = 4, r = 3 has virtual birack matrix

Definition 4 A virtual birack labeling of an oriented blackboard framed virtual link diagram L by a virtual birack X, also called an X-labeling of L, is an assignment of an element of X to each semiarc in L such that

the conditions



are satisfied at every crossing.

The virtual birack axioms are the result of translating the the oriented blackboard framed virtual Reidemeister moves into conditions on labelings of semiarcs using the labeling rule in definition 4. Invertibility of B, V and existence, uniqueness and invertibility of S and vS encode the Reidemeister II and vII moves; diagonal invertibility of B satisfies the framed I move, while the requirement that V fixes the diagonal satisfies the vI move. The Yang-Baxter equations then encode the Reidemeister III, vIII and v moves. Indeed, by construction we have:

Theorem 1 If X is a virtual birack and L and L' are virtual link diagrams related by blackboard framed virtual Reidemeister moves, then there is a bijection between the sets of virtual birack labelings of L and L'.

Definition 5 Let L be a blackboard framed virtual link diagram and X a finite virtual birack. The cardinality of the set of X-labelings of L, denoted $\Phi_X^B(L)$, is the basic virtual birack counting invariant of L with respect to X.

Note that $\Phi_X^B(L) = |\text{Hom}(VB(L), X)|$ where Hom(VB(L), X) is the set of virtual birack homomorphisms from the fundamental virtual birack VB(L) of L to the labeling birack X.

Let X be a virtual birack. The maps $\alpha, \pi: X \to X$ defined by $\alpha = (S^{-1} \circ \Delta)_2^{-1}$ and $\pi = (S^{-1} \circ \Delta)_1 \circ \alpha$ give the virtual birack labels on the semiarcs in a framed type I move:



Since π represents going through a positive kink, π is known as the kink map of the virtual birack X. The order of π , i.e. the smallest positive integer N such that $\pi^N = \text{Id} : X \to X$, is the birack rank or birack characteristic of X.

Virtual birack labelings of a virtual knot or link are preserved by blackboard framed virtual Reidemeister moves but not in general by Reidemeister I moves. However, if a virtual birack X has finite birack rank N, then labelings of X are preserved by N-phone cord moves:



In particular, if L and L' are related by framed virtual Reidemeister and N-phone cord moves, then $\Phi_X^B(L) = \Phi_X^B(L')$. See [21] for more.

Thus, if L is an unframed oriented virtual link of c components, there is a c-dimensional lattice of framings of L corresponding to writhe vectors in \mathbb{Z}^c , each representing a distinct framed link. We therefore have, associated to an unframed oriented virtual link, a \mathbb{Z}^c -lattice of basic birack counting invariants. The fact that N-phone cord moves preserve $\Phi^B_X(L)$ then implies that this lattice is tiled with repeats of a c-dimensional tile corresponding to $(\mathbb{Z}_N)^c$. In particular, the sum over one tile of the basic counting invariants yields an invariant of unframed oriented virtual links known as the *integral virtual birack counting invariant*.

Definition 6 Let X be a finite virtual birack of birack rank N and let L be an unframed oriented virtual link of c components. For every $\mathbf{w} \in (\mathbb{Z}_N)^c$, let (L, \mathbf{w}) be a diagram of L with framing vector \mathbf{w} . Then

$$\Phi^{\mathbb{Z}}_X(L) = \sum_{\mathbf{w} \in (\mathbb{Z}_N)^c} \Phi^B_X(L, \mathbf{w})$$

is the integral virtual birack counting invariant.

For instance, if N = 2 then for the virtual Hopf link we have



By construction and theorem 1, we have

Theorem 2 If X is a finite virtual birack and L and L' are virtually isotopic oriented virtual link diagrams, then $\Phi_X^{\mathbb{Z}}(L) = \Phi_X^{\mathbb{Z}}(L')$.

Example 3 Let X be the virtual birack with underlying set $\{1, 2\}$ and virtual birack matrix

$$M_X = \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right].$$

This matrix describes the labelings of a virtual link diagram with elements of X by the following rules: when going through a classical undercrossing, semiarc labels stay the same; at classical overcrossings and at virtual crossings from either direction, semiarc labels switch from 1 to 2 or from 2 to 1. The kink map here is the transposition $\pi = (12)$, which has order N = 2; thus, for any link L, we must consider a complete set of diagrams with writhes mod 2 on each component. For example, the virtual Hopf link has c = 2 components and thus we have $N^c = 2^2 = 4$ diagrams, corresponding to writhe vectors in $(\mathbb{Z}_N)^c = (\mathbb{Z}_2)^2 = \{(0,0), (1,0), (0,1), (1,1)\}$ in a complete tile of framings mod 2. It is not hard to see that, for instance, the $\mathbf{w} = (0,0)$ framing has no valid labelings by X:



It turns out that there are four valid labelings, all in the $\mathbf{w} = (1,0)$ framing. Thus, we have $\Phi_x^{\mathbb{Z}}(L) = 4$.



2.3 Virtual birack shadows

Shadow labelings, that is, labelings of the regions between the arcs of a knot diagram, have been used in connection with rack, quandle, birack and biquandle counting invariants in [6, 10, 22]. We now extend this idea to the case of virtual biracks.

Definition 7 Let X be a virtual birack and S a set. A virtual birack shadow structure or X-shadow structure on S is a right invertible right action of X on S (i.e. a map $\cdot : S \times X \to S$) satisfying for all $x, y \in X$ and $A \in S$:

$$(A \cdot y_x) \cdot x^y = (A \cdot y_{\widetilde{x}}) \cdot x^{\widetilde{y}} = (A \cdot x) \cdot y$$

Elements of an X-shadow S are used to label the regions between the strands in an X-labeled virtual link diagram:

Definition 8 Let X be a virtual birack, S an X-shadow, and L a blackboard framed oriented virtual link diagram. Then a *shadow labeling* or X, S-labeling of L is an assignment of elements of X to the semiarcs in L and elements of S to regions between the semiarcs of L such that the virtual birack labeling conditions are satisfied at every crossing and at every semiarc we have



The shadow axioms are chosen to guarantee that shadow labelings are well-defined at classical and virtual crossings. It is straightforward to show that the requirement that shadow labels are well-defined around classical and virtual crossings is sufficient to guarantee that shadow labelings are preserved under Reidemeister moves. Moreover, well-definedness of shadow labels around positive crossings implies well-

definedness around negative crossings.



Let $X = \{x_1, \ldots, x_n\}$ be a virtual birack and $S = \{A_1, \ldots, A_m\}$ an X-shadow. We can specify the shadow operation with an $m \times n$ matrix M_S whose row *i* column *j* entry is *k* such that $A_i \cdot x_j = A_k$. Note that right-invertibility of \cdot requires the columns of M_S to be permutations, and the columns corresponding to birack elements *x* and $\pi(x)$ are the same.

Example 4 Let X be a virtual birack, S any set and $\sigma : S \to S$ any bijection. Then $A \cdot x = \sigma(A)$ defines an X-shadow structure on S, called a *constant action shadow*: bijectivity of σ gives us right-invertibility, and we have for any $x, y \in X$

$$(A \cdot y_x) \cdot x^y = (A \cdot y_{\widetilde{x}}) \cdot x^{\widetilde{y}} = (A \cdot x) \cdot y = \sigma^2(x).$$

In particular, if $S = \{A, B\}$ and every column is the transposition (AB), then X, S-labelings are called *checkerboard colorings* of L.

Definition 9 Let X be a virtual birack with birack rank N and S an X-shadow. For any blackboard framed oriented virtual link L of c components, let $\mathcal{L}(L, X, S)$ be the set of X, S-labelings of L. Then set

$$\Phi_{X,S}^{\mathbb{Z}}(L) = \sum_{w \in (\mathbb{Z}_n)} |\mathcal{L}((L, \mathbf{w}), X, S)|.$$

It is apparent that the number of shadow labelings for a fixed X-labeling is equal to the cardinality of S, since a choice of shadow label for one "source" region determines shadow labels for all other regions, and there are |S| such choices. Hence, we have

Theorem 3 Let X be a finite birack. For any X-shadow and oriented virtual link L, we have $\Phi_{X,S}^{\mathbb{Z}}(L) = |S| \Phi_X^{\mathbb{Z}}(L)$.

Corollary 4 Let X be a virtual birack and S an X-shadow. If L and L' are virtually isotopic virtual links, then we have $\Phi_{X,S}^{\mathbb{Z}}(L) = \Phi_{X,S}^{\mathbb{Z}}(L')$.

Example 5 Let X be the virtual birack from example 3 and let S be the checkerboard coloring shadow, i.e. the set $S = \{A, B\}$ with shadow matrix

$$M_S = \left[\begin{array}{cc} B & B \\ A & A \end{array} \right].$$

Then over a complete period of framings mod 2, there are eight shadow labelings of the virtual Hopf link from example 3, i.e. $\Phi_{X,S}^{\mathbb{Z}}(L) = 8$.

2.4Virtual Birack Shadow Algebras and Modules

The virtual shadow counting invariant is an enhancement of the integral virtual birack counting invariant, but an enhancement that is equivalent to the original unenhanced invariant. To get a stronger enhancement, we need additional algebraic structure.

Definition 10 Let X be a set with virtual birack structure $B(x, y) = (y^x, x_y)$ and $V(x, y) = (y^{\hat{x}}, x_{\hat{y}})$ and let S be an X-shadow. Then the virtual birack shadow algebra $\mathbb{Z}[X,S]$ is the quotient of the polynomial algebra $\mathbb{Z}[v_{A,x,y}^{\pm 1}, t_{A,x,y}^{\pm 1}, s_{A,x,y}, r_{A,x,y}^{\pm 1}]$ for $A \in S, x, y \in X$ modulo the ideal I generated by elements of the form

- $t_{A,x_{z}y,x_{z},x_{z}y}s_{A,y_{z},x_{z},z} s_{A,x_{yz},y}x_{z}x_{y}t_{A,z,x,y}$ $t_{A,y_{z},x,zy}s_{A,y,z} s_{A,x_{yz},y}x_{z}x_{y}t_{A,z,x,y}$ $s_{A,y_{z},x,zy} t_{A,x_{yz},y}x_{z}x_{y}s_{A,x_{y},z}r_{A,z,x,y} s_{A,x_{yz},y}x_{z}x_{y}s_{A,z,x,y}$, and $1 \prod_{k=0}^{N-1} (t_{A,-1}\alpha(\pi^{k}(x)),\pi^{k}(x),\alpha(\pi^{k}(x))r_{A,-1}\alpha(\pi^{k}(x)),\pi^{k}(x),\alpha(\pi^{k}(x)) + s_{A,-1}\alpha(\pi^{k}(x)),\pi^{k}(x),\alpha(\pi^{k}(x)))$

The virtual birack algebra is motivated by the (v, t, s, r)-virtual birack definition; given an X, S-labeled link diagram, we define a secondary labeling of the semiarcs by beads which obey (v, t, s, r)-style relations with coefficients which depend on the X, S labels at the crossing. The relations are obtained from the framed virtual Reidemeister moves and the N-phone cord move using the bead relations



For instance, the virtual move vIII yields the requirements



Repeating for the other framed virtual Reidemeister moves yields the relations in definition 10.

Definition 11 A virtual shadow module or $\mathbb{Z}[X, S]$ -module is a representation of $\mathbb{Z}[X, S]$, i.e., an abelian group G with a family of automorphisms $v_{A,x,y}, t_{A,x,y}, r_{A,x,y} : G \to G$ and endomorphisms $s_{A,x,y} : G \to G$ for $A \in S, x, y \in X$ such that each generator of the ideal I in definition 10 is the zero map.

Example 6 Let G be a commutative ring. For any X-shadow $S = \{A_1, A_2, \ldots, A_m\}$ we can give G the structure of a $\mathbb{Z}[X, S]$ -module by choosing invertible elements $v_{A,x,y}, t_{A,x,y}, r_{A,x,y} \in G^*$ and elements $s_{A,x,y} \in G$ for each $A \in S$ and $x, y \in X$ such that the ideal $I \subset G$ in definition 10 is zero. We can express such a structure with an $m \times 4$ block matrix of $|X| \times |X|$ blocks

	V_{A_1}	T_{A_1}	S_{A_1}	R_{A_1}
	V_{A_2}	T_{A_2}	S_{A_2}	R_{A_2}
$M_G =$:	•	•	:
	V_{A_m}	T_{A_m}	S_{A_m}	R_{A_m}

where the row j column k entry of T_{A_i} is t_{A_i,x_j,y_k} , etc.

Example 7 Let X be a virtual birack and S an X-shadow. For any X, S-labeling f of a blackboard framed oriented link diagram L, there is an X, S-module generated by beads associated to semiarcs in L with relations determined by crossings, called the *fundamental* X, S-module of the labeling f of L, denoted $\mathbb{Z}[f]$. For instance, the X, S-labeling of the virtual Hopf link diagram below where X, S are as in example 3 has fundamental $\mathbb{Z}[X,S]$ -module $\mathbb{Z}[f]$ with presentation matrix $M_{\mathbb{Z}[f]}$ below.



Example 8 Let X be the virtual birack on one element and S the X-shadow with one element. Then there is a unique X, S-labeling of any virtual link L, and its fundamental X, S-module of virtual shadow module over the ring $G = \mathbb{Z}[t^{\pm 1}, r^{\pm 1}]$ with matrix $M_G = \begin{bmatrix} 1 | t | 1 - tr | r \end{bmatrix}$ is known as the generalized Alexander module of L; the determinant of the matrix presenting $M_{\mathbb{Z}[f]}$ is a two-variable Laurent polynomial invariant of virtual links known as the Generalized Alexander polynomial. See [20, 24] for more.

Definition 12 Let L be a virtual link of c components, X a virtual birack of birack rank N, S an X-shadow and G an abelian group with the structure of a $\mathbb{Z}[X,S]$ -module. Let $\mathcal{L}((L,\mathbf{w}), X, S)$ be the set of X, S-labelings of a diagram of L with writhe vector $\mathbf{w} \in (\mathbb{Z}_N)^c$. Then the virtual shadow module multiset of L with respect to G is the multiset of G-modules

$$\Phi_{X,S}^{M,G}(L) = \{ \operatorname{Hom}_{G}(\mathbb{Z}[f], G) : f \in \mathcal{L}((L, \mathbf{w}), X, S), \mathbf{w} \in (\mathbb{Z}_{N})^{c} \}$$

and the virtual shadow module polynomial of L with respect to G is

$$\Phi_{X,S}^G(L) = \sum_{\mathbf{w} \in (\mathbb{Z}_N)^c} \left(\sum_{f \in \mathcal{L}((L,\mathbf{w}),X,S)} u^{|\operatorname{Hom}_G(\mathbb{Z}[f],G)|} \right).$$

By construction, we have

Theorem 5 If L and L' are virtually isotopic virtual links, then $\Phi_{X,S}^{M,G}(L) = \Phi_{X,S}^{M,G}(L')$ and $\Phi_{X,S}^{G}(L) = \Phi_{X,S}^{G}(L')$.

Example 9 Let X, S be the virtual birack and shadow from example 5. Let $G = \mathbb{Z}_5$; then thinking of G as a commutative ring as in example 6, we can give G the structure of a $\mathbb{Z}[X, S]$ -module with the shadow module matrix

$$M_G = \begin{bmatrix} 2 & 2 & 1 & 4 & 3 & 4 & 1 & 1 \\ 2 & 2 & 1 & 4 & 1 & 3 & 1 & 1 \\ \hline 1 & 1 & 2 & 3 & 3 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 & 4 & 3 & 2 & 2 \end{bmatrix}$$

To find the bead labelings of the X, S-labeling of the virtual Hopf link from example 7, we replace the entries in the matrix $M_{\mathbb{Z}[f]}$ presenting the fundamental $\mathbb{Z}[X, S]$ -module of f with their values in M_G to obtain a matrix over \mathbb{Z}_5 whose solution space yields the set of bead labelings of the semiarcs with beads in \mathbb{Z}_5 :

0	4	2	4	0	0		1	0	0	0	0	0]
0	2	4	0	0	0		0	1	0	0	0	0
4	0	0	1	1	0		0	0	1	0	0	0
0	0	0	0	1	4	\rightarrow	0	0	0	1	0	0
0	0	0	0	4	2		0	0	0	0	1	0
3	4	0	0	0	0		0	0	0	0	0	1

Thus, X, S-labeling in example 5 has only the all-zeroes bead labeling with respect to our chosen M_R , and this labeling contributes u^1 to the value of $\Phi_{X,S}^G(L)$. Repeating for all eight shadow labelings, we get $\Phi_{X,S}^G(L) = 8u$.

Example 10 Repeating the computation from example 9 for the unlink of two components and the classical Hopf link with the same values of X, S and M_G , we obtain



In particular, this example shows that $\Phi_{X,S}^G$ is not determined by $\Phi_{X,S}^{\mathbb{Z}}$, since all three links have $\Phi_{X,S}^{\mathbb{Z}} = 8$.

3 Twisted Virtual Shadow Algebras and Modules

3.1 Twisted virtual knots and links

Twisted virtual knots and links were introduced in [4] in 2008, generalizing the notion of virtual knots as knots in *I*-bundles over compact orientable surfaces to the case of compact nonorientable surfaces. Classical crossings in a twisted virtual link diagram correspond to crossings in $\Sigma \times I$, while virtual crossings correspond to genus in $\Sigma \times I$ and *twist bars* on strands indicate when a strand has traversed a cross-cap in Σ .



A twisted oriented blackboard framed virtual link is an equivalence class of twisted oriented virtual link diagrams, i.e. oriented virtual link diagrams with twist bars, under the equivalence relation generated by the virtual Reidemeister moves together with the *twisted moves*:



Remark 1 Note that while a twist bar can always move past a virtual crossing, in general it cannot move past a classical crossing. A twist bar *can* be pushed through a classical kink, so the blackboard framing given by writhe numbers for each component is still well defined in the twisted virtual setting. See [4, 9] for more.

3.2 Twisted virtual biracks

Adding twists to our set of allowed moves adds a map $T: X \to X$ to our algebraic structure defined by semiarc labelings. Specifically, in a twisted virtual link diagram, semiarcs in a twisted virtual link diagram are obtained by dividing the link at classical over and undercrossings, virtual crossings and twist bars. As before, we will refer to a portion of a twisted virtual link diagram between classical crossing points as a *classical semiarc*. Our semiarc labeling rule now includes:



Definition 13 Let X be a set and let $\Delta : X \to X \times X$ be the diagonal map $\Delta(x) = (x, x)$. A twisted virtual birack structure on X is a pair of invertible maps $B, V : X \times X \to X \times X$ and an involution $T : X \to X$ satisfying

(i) X is a virtual birack with operations B and V,

$$(T \times \mathrm{Id}_X)V = V(\mathrm{Id}_X \times T)$$
 and $(\mathrm{Id}_X \times T)V = V(T \times \mathrm{Id}_X)$, and

(iii)

$$(T \times T)B(T \times T) = VBV.$$

If we also have $(S \circ \Delta)_1 = (S \circ \Delta)_2$, X is a twisted virtual biquandle. See [9, 17] for more.

Example 11 Let $\overline{\Lambda} = \mathbb{Z}[t^{\pm 1}, r^{\pm 1}, v^{\pm 1}, T]/(1 - T^2, t - v^2 r)$. Then any $\overline{\Lambda}$ -module is a twisted virtual birack under the operations $B(x, y) = (ty, rx), V(x, y) = (vy, v^{-1}x), T(x) = Tx$. See [9] for more.

As with virtual biracks, we can represent a twisted virtual birack structure on a set $X = \{x_1, \ldots, x_n\}$ with a matrix (in this case, $n \times (4n+1)$) encoding the operation tables of the $x^y, x_y, x^{\widetilde{y}}, x_{\widetilde{y}}$ and T(x) operations:

$$M_X = \left[\begin{array}{c|c} B_1 & B_2 & V_1 & V_2 & T \end{array} \right]$$

with $(B_1)_{j,i} = k$, $(B_2)_{i,j} = l$, $(V_1)_{j,i} = m$, $(V_2)_{i,j} = p$ and $T_i = q$ such that $B(x_i, x_j) = (x_k, x_l)$, $V(x_i, x_j) = (x_m, x_p)$ and $T(x_i) = x_q$.

A twisted virtual birack is a virtual birack; more precisely, there is a forgetful functor from the category of twisted virtual biracks to the category of virtual biracks defined by forgetting the twist operation, i.e.,

 $\mathcal{F} : \mathbf{Tvb} \to \mathbf{Vb}$ by $\mathcal{F}(X, B, V, T) = (X, B, V).$

However, not every virtual birack has a compatible twisted structure $T: X \to X$:

Theorem 6 The virtual birack X with virtual birack matrix

$$M_X = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ \end{array} \right]$$

has no twisted structure $T: X \to X$ satisfying all of the twisted virtual birack axioms.

Proof. A twisted structure $T : X \to X$ is a transposition, so there are only two possible twist maps: $T = \text{Id}_X$ and T = (12). Then for both $T = \text{Id}_X$ and T = (12), we have

$$(T \times T)B(T \times T)(x, y) = (y, (12)x) \neq ((12)y, x) = VBV(x, y).$$

3.3 Twisted virtual birack shadows

Definition 14 Let X be a twisted virtual birack. A *twisted virtual birack shadow* is a virtual birack shadow S over X considered as a virtual birack such that for every $x \in X$ we have $A \cdot Tx = A \cdot x$.



As before, for any twisted virtual birack X of rack rank N and X-shadow S, we have an invariant of twisted virtual links

$$\Phi_{X,S}^{\mathbb{Z}}(L) = \sum_{w \in (\mathbb{Z}_n)} |\mathcal{L}((L, \mathbf{w}), X, S)|$$

where $\mathcal{L}((L, \mathbf{w}), X, S)$ is the set of X, S-labelings of a diagram of L with writh vector **w**. Moreover, we have

Theorem 7 Let X be a twisted virtual birack and S an X-shadow. Then for any unframed oriented twisted virtual link L, $\Phi_{X,S}^{\mathbb{Z}}(L) = |S| \Phi_X^{\mathbb{Z}}(L).$

Corollary 8 Let X be a virtual birack and S an X-shadow. If L and L' are virtually isotopic virtual links, then we have $\Phi_{X,S}^{\mathbb{Z}}(L) = \Phi_{X,S}^{\mathbb{Z}}(L').$

$\mathbf{3.4}$ Twisted virtual shadow algebras

As in the virtual birack case, we use a secondary labeling by beads to enhance $\Phi_{X,S}^{\mathbb{Z}}(L)$.

Definition 15 Let X be a set with twisted virtual birack structure $B(x, y) = (y^x, x_y), V(x, y) = (y^{\widetilde{x}}, x_{\widetilde{y}})$ and $T: X \to X$ and let S be an X-shadow. Then the twisted virtual shadow algebra $\mathbb{Z}[X,S]$ is the quotient of the polynomial algebra $\mathbb{Z}[v_{A,x,y}^{\pm 1}, t_{A,x,y}^{\pm 1}, r_{A,x,y}^{\pm 1}, q_{A,x}]$ for $A \in S, x, y \in X$ modulo the ideal I generated by elements of the form

• $v_{A,y_{\widetilde{x}},x^{\widetilde{y}}} - v_{A,x,y}$

•
$$v_{A \cdot (x_{\widetilde{y}})_{\widetilde{z}}, y^{\widetilde{x}}, z^{\widetilde{x}_{\widetilde{y}}}} v_{A, x_{\widetilde{z}}, y_{\widetilde{z}}} - v_{A \cdot z, x, y} v_{A, y, z}$$

•
$$v_{A \cdot y_z, x, z^y} \iota_{A, y, z} - \iota_{A \cdot (x_{\widetilde{y}})_{\widetilde{z}}, y^{\widetilde{x}}, z^{\widetilde{x}_{\widetilde{y}}}} v_{A, x_{\widetilde{y}}, z}$$

- $v_{A,Tx,y} v_{A,x,y}$
- $v_{A,x,Ty}q_{A,y} q_{A \cdot x_{\widetilde{u}},y^{\widetilde{x}}}v_{A,x,y}$

- $v_{A \cdot y_{\widetilde{z}}, x, z^{\widetilde{y}}} v_{A, y, z} v_{A \cdot (x_{\widetilde{y}})_{\widetilde{z}}, y^{\widetilde{x}}, z^{\widetilde{x}_{\widetilde{y}}}} v_{A, x, y}$
- $v_{A\cdot z,x,y}v_{A,x_{\widetilde{y}},z} v_{A\cdot y_{\widetilde{z}},x,z^{\widetilde{y}}}v_{A,x_{\widetilde{z}},y_{\widetilde{z}}}$
- - $q_{A\cdot y,x}v_{A,x,y} v_{A,Tx,y}q_{A,x_{\widetilde{y}}}$

•
$$v_{A,(x_{\widetilde{y}})}(y_{\widetilde{x}})q_{A,(Tx)}(Ty)}r_{A,T_x,T_y}q_{A\cdot y,x}v_{A,x,y}-t_{A,y^{\widetilde{x}},x_{\widetilde{y}}}$$

- $q_{A\cdot(Tx)_{(Ty)},(Ty)}(Tx)t_{A,Tx,Ty}q_{A,y} v_{A,(x_{\widetilde{y}})}(y_{\widetilde{x}})r_{A,y^{\widetilde{x}},x_{\widetilde{y}}}v_{A,x,y}$ and $1 \prod_{k=0}^{N-1} t_{A\cdot^{-1}\alpha(\pi^{k}(x)),\pi^{k}(x),\alpha(\pi^{k}(x))}r_{A\cdot^{-1}\alpha(\pi^{k}(x)),\pi^{k}(x),\alpha(\pi^{k}(x))}$

As in the virtual birack case, the twisted virtual shadow algebra relations come from the oriented framed twisted virtual Reidemeister moves where we interpret the generators $v_{A,x,y}$, $t_{A,x,y}$, $r_{A,x,y}$ and $q_{A,x}$ as coefficients for beads on semiarcs as pictured:

Then for instance, move tv requires that



Remark 2 If $s_{A,x,y}$ coefficients are included in the bead operations at classical crossings as in the virtual birack algebra case, the tv requires that $s_{A,x,y} = 0$ for all $A \in S, x, y \in X$.

Definition 16 Let X be a twisted virtual birack and S an X-shadow. A twisted virtual birack shadow module or $\mathbb{Z}[X, S]$ -module is a representation of $\mathbb{Z}[X, S]$, i.e., an abelian group G with automorphisms $v_{A,x,y}, t_{A,x,y}, r_{A,x,y}, q_{A,x}$ for $A \in S$, $x, y \in X$ such that the maps generating the ideal I in definition 15 are zero.

As before, an X, S-labeling f of an oriented twisted virtual link diagram defines a fundamental $\mathbb{Z}[X, S]$ module $\mathbb{Z}[f]$ with presentation matrix M_f expressing the system of linear equations determined beads on the semiarcs.

Definition 17 Let L be a virtual link of c components, X a twisted virtual birack of birack rank N, S an X-shadow and G an abelian group with the structure of a $\mathbb{Z}[X,S]$ -module. Let $\mathcal{L}((L,\mathbf{w}), X, S)$ be the set of X, S-labelings of a diagram of L with writhe vector $\mathbf{w} \in (\mathbb{Z}_N)^c$. Then the *twisted virtual shadow module multiset* of L with respect to G is the multiset of G-modules

$$\Phi_{X,S}^{M,G}(L) = \{ \operatorname{Hom}_G(\mathbb{Z}[f], G) : f \in \mathcal{L}((L, \mathbf{w}), X, S), \mathbf{w} \in (\mathbb{Z}_N)^c \}$$

and the twisted virtual shadow module polynomial of L with respect to G is

$$\Phi_{X,S}^G(L) = \sum_{\mathbf{w} \in (\mathbb{Z}_N)^c} \left(\sum_{f \in \mathcal{L}((L,\mathbf{w}),X,S)} u^{|\operatorname{Hom}_G(\mathbb{Z}[f],G)|} \right)$$

By construction, we have

Theorem 9 If L and L' are twisted virtually isotopic oriented twisted virtual links, then $\Phi_{X,S}^{M,G}(L) = \Phi_{X,S}^{M,G}(L')$ and $\Phi_{X,S}^{G}(L) = \Phi_{X,S}^{G}(L')$.

4 Examples and Applications

In this section we collect some examples and applications of the new enhanced invariants.

Example 12 For our first application, let X, S be the virtual birack and shadow from example 5 and consider the $\mathbb{Z}[X, S]$ -module structure on $G = \mathbb{Z}_5$ given by the virtual birack shadow module matrix

$$M_G = \begin{bmatrix} 1 & 1 & 1 & 1 & 4 & 1 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 4 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 & 4 & 1 & 4 & 4 \\ 2 & 2 & 2 & 2 & 1 & 4 & 4 & 4 \end{bmatrix}$$

The oriented virtual knot 4.4 has $\Phi_{X,S}^G(4.4) = 4u$ while its reverse $\overline{4.4}$ has $\Phi_{X,S}^G(\overline{4.4}) = 4u^5$. In particular, since reversing the orientation of a virtual knot yields an isomorphic knot group, the invariants $\Phi_{X,S}^G$ are not determined by the isomorphism type of the knot group.



Example 13 Slavik's knot is not detected by the arrow polynomial, and the Miyazawa knot is not detected by the Miyazawa polynomial (see [11]). However, both are distinguished from the unknot and from each other by $\Phi_{X,S}^G$ where X, S are as in example 7, $G = \mathbb{Z}_5$ and the $\mathbb{Z}[X, S]$ -module structure on G is given by the matrix:



Example 14 For our next example, we randomly a selected $\mathbb{Z}[X, S]$ -module structure on $G = \mathbb{Z}_5$ for the virtual birack X and X-shadow S from example 5 and computed $\Phi_{X,S}^G$ for the virtual knots in the knot atlas [2] using our custom python code, available at www.esotericka.org.

$G = \mathbb{Z}_5,$	$M_G =$	$\begin{bmatrix} 2 \end{bmatrix}$	2	2	3	4	2	2	2
		2	2	2	3	3	4	2	2
		4	4	1	4	4	3	1	1
		4	4	1	4	2	4	1	1

 $\Phi^G_{X,S}(L) \mid L$

4u	2.1, 3.2, 3.3, 3.4, 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.9, 4.11, 4.12, 4.13, 4.14, 4.15, 4.17, 4.18, 4.19, 4.20, 4.21, 4.22, 4.22, 4.2
	4.23, 4.24, 4.25, 4.26, 4.27, 4.28, 4.29, 4.31, 4.33, 4.34, 4.35, 4.36, 4.37, 4.38, 4.39, 4.40, 4.42, 4.43, 4.43, 4.44
	4.44, 4.45, 4.46, 4.48, 4.49, 4.51, 4.52, 4.54, 4.57, 4.60, 4.61, 4.62, 4.63, 4.64, 4.65, 4.66, 4.67, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.73, 4.78, 4.69, 4.69, 4.69, 4.69, 4.69, 4.69, 4.69, 4.69, 4.69, 4.69, 4.69, 4.69, 4.73, 4.78, 4.78, 4.69, 4.78, 4.69, 4.78
	4.79, 4.80, 4.81, 4.82, 4.83, 4.84, 4.87, 4.88, 4.89, 4.92, 4.93, 4.94, 4.95, 4.97, 4.101, 4.103, 4.104
$4u^5$	3.1, 3.5, 3.6, 3.7, 4.7, 4.8, 4.10, 4.16, 4.30, 4.32, 4.41, 4.47, 4.50, 4.53, 4.55, 4.56, 4.58, 4.59, 4.68, 4.59, 4.68, 4.59, 4.68, 4.59, 4.68, 4.59
	4.70, 4.72, 4.74, 4.75, 4.76, 4.77, 4.85, 4.86, 4.90, 4.91, 4.96, 4.984.100, 4.102, 4.106, 4.107, 4.108
$4u^{25}$	4.71, 4.99, 4.105

Example 15 For our final example, we demonstrate that the twisted virtual shadow module invariant Φ_{XS}^G is not determined by the twisted Jones polynomial defined in [4]. The twisted virtual links below both have twisted Jones polynomial $(-A^{-2} - A^{-4})(-A^{-2} - A^2)$ but are distinguished by Φ_{XS}^G with the twisted virtual birack X, trivial X-shadow structure on $S = \{A\}$ and $\mathbb{Z}[X, S]$ -module structure on $G = \mathbb{Z}_3$ below:

5 Questions

For simplicity, we have limited ourselves to computing $\Phi_{X,S}^G$ in the case when G is a commutative ring – specifically, the case where $G = \mathbb{Z}_n$. We expect that $\Phi_{X,S}^G$ should be even stronger if we expand to the case of noncommutative rings, e.g. $n \times n$ matrices over \mathbb{Z}_n .

In the biquandle case, we can define X-labelings of semisheets in abstract knotted surface diagrams. What new or different relations, if any, are imposed by the Roseman moves on the set of bead labelings? That is, define virtual surface biquandle algebras and modules.

How do the enhancement strategies of the quandle counting invariant from [5] apply in the case of virtual and twisted virtual shadow algebras?

What is the relationship of $\Phi_{X,S}^G$ to birack cocycle invariants?

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