Towards Directed Collapsibility (Research)

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1 Introduction

Spaces that are equipped with a direction have only recently been given more attention from a topological point of view. The spaces of directed paths are the defining feature for distinguishing different directed spaces. One reason for studying directed spaces is their application to the modeling of concurrent programs, where

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standard algebraic topology does not provide the tools needed [4]. Concurrent programming is used when multiple processes need to access shared resources. Directed spaces are models for concurrent program, where paths respecting the time directions represent executions of programs. In such models, executions are equivalent if their execution paths are homotopic through a family of directed paths. This observation has already led to new insights and algorithms. For instance, verification of concurrent programs is simplified by verifying one execution from each connected component of the space of directed paths; see [4, 5].

While equivalence of executions is clearly stated in concurrent programming, equivalence of the directed topological spaces themselves is not well understood. Directed versions of homotopy groups and homology groups are not agreed upon. Directed homeomorphism is too strong; whereas, directed homotopy equivalence is often too weak, to preserve the properties of the concurrent programs. In classical (undirected) topology, the concept of simplifying a space by a sequence of collapses goes back to J.H.C. Whitehead [11], and has been studied in [1, 6], among others. However, a definition for a directed collapse of a Euclidean cubical complex that preserves spaces of directed paths is notably missing from the literature.

In this article, we consider spaces of directed paths in Euclidean cubical complexes. Our objects of study are spaces of directed paths relative to a fixed pair of endpoints. We show how local information of the past links of vertices in a Euclidean cubical complex can provide global information on the spaces of directed paths. As an example, our results are applied to study the spaces of directed paths in the well-known dining philosophers problem. Furthermore, we define directed collapse so that a directed collapse of a Euclidean cubical complex preserves the relevant spaces of directed paths in the original complex. Our theoretical work has applications to simplifying verification of concurrent programs without loops, and better understanding partial executions in those concurrent programs.

We begin, in Sect. 2, with two motivating examples of how the execution of concurrent programs can be modeled by Euclidean cubical complexes and directed path spaces. In Sect. 3, we introduce the notions of spaces of directed paths and Euclidean cubical complexes. Given the directed structure of these Euclidean cubical complexes, we do not consider the link of a vertex but the *past* link of it. In Sect. 4, we give results on the topology of the spaces of directed paths from an initial vertex to other vertices in terms of past links. Theorem 1 gives sufficient conditions on the past links of every vertex of a complex so that spaces of directed paths are contractible. Theorem 2 gives conditions that are sufficient for the spaces of directed paths to be connected. In Theorem 3, we give sufficient conditions on the past link of a vertex so that the space of directed paths from the initial vertex to that vertex is disconnected. In Sect. 5, we describe a method of collapsing one complex into a simpler complex, while preserving the directed path spaces.

2 Concurrent Programs and Directed Path Spaces

We illustrate how to organize possible executions of concurrent programs using Euclidean cubical complexes and directed spaces. An execution is a scheduling of the events that occur in a program in order to compute a specific task. In Example 1, we describe the dining philosophers problem. In Example 2, we illustrate how to model executions of concurrent programs in the context of the dining philosophers problem in the case of two philosophers.

Example 1 (Dining Philosophers) The dining philosophers problem originally formulated by E. Dijkstra [2] and reformulated by T. Hoare [7] illustrates issues that arise in concurrent programs. Consider n philosophers sitting at a round table ready to eat a meal. Between each pair of neighboring philosophers is a chopstick for a total of n chopsticks. Each philosopher must eat with the two chopsticks lying directly to the left and right of her. Once the philosopher is finished eating, she must put down both chopsticks. Since there are only n chopsticks, the philosophers must share the chopsticks in order for all of them to eat. The dining philosopher problem is to design a concurrent program where all n philosophers are able to eat once for some finite amount of time.

A design of a program is a choice of actions for each philosopher. One example of a design of a program is where each of the n philosophers does the following:

1. Wait until the right chopstick is available, then pick it up.
2. Wait until the left chopstick is available, then pick it up.
3. Eat for some finite amount of time.
4. Put down the left chopstick.
5. Put down the right chopstick.

While correct executions of this program are possible (e.g., where the philosophers take turns eating alone), this design has states in which every philosopher has picked up the chopstick to her right and is waiting for the other chopstick. Such a situation exemplifies a *deadlock* in concurrent programming, an execution that gets "stuck" and never finishes.

The design described above also has states that cannot occur. For example, consider the dining philosophers problem when $n = 2$. The state in which both philosophers are finished eating and one is still holding onto chopstick a while the other is holding chopstick b would imply that a philosopher was able to eat with only one chopstick—an example of an *unreachable* state in concurrent programming.

The dining philosophers problem illustrates the difficulties in designing concurrent programs. Difficulties arise since each philosopher must use chopsticks that must be shared with the neighboring philosophers. Analogously, in concurrent programming, multiple processes must access shared resources that have a finite capacity.

The next example illustrates how to model executions of the dining philosophers problem with a Euclidean cubical complex. When the problem consists of two

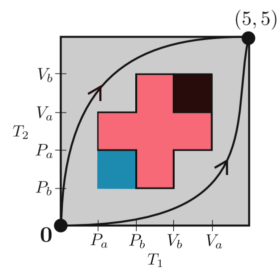


Fig. 1 The Swiss Flag. The pink region is the forbidden region. Any bi-monotone path outside of F is a possible execution. The set of all executions of two processes, T_1 and T_2, is called the *state-space* Two regions in the state space are of particular interest. The black region is the set of all unreachable states, and the blue region is the set of all states that are doomed to never complete. A state is doomed if any path starting at that state leads to a deadlock. The black curves in the figure are two possible paths in this directed space

philosophers, the Euclidean cubical complex used to model the dining philosophers problem is often referred to as the *Swiss Flag*.

Example 2 (Swiss Flag) In the language of concurrent programming, the two philosophers represent two processes denoted by T_1 and T_2. The two chopsticks represent shared resources denoted by a and b. One process is executing the program $P_a P_b V_b V_a$ and the other process is executing the program $P_b P_a V_a V_b$. Here, P means that a process has a *lock* on that resource while V means that a process *releases* a resource. To model this concurrent program with a Euclidean cubical complex, we construct a 5×5 grid where the x-axis is labeled by $P_a P_b V_b V_a$, each a unit apart, and the y-axis is labeled by $P_b P_a V_a V_b$, each also a unit apart (see Fig. 1). The region $[1, 4] \times [2, 3]$ represents when both T_1 and T_2 have a lock on a. In the dining philosophers problem, a single chopstick can only be held by one philosopher at a given time. The mutual exclusion of the chopsticks translates to the shared resources, a and b, each having *capacity* one, where the capacity of a resource is the number of processes that can have access to the resource simultaneously. We call the region $[1, 4] \times [2, 3]$ *forbidden* since T_1 and T_2 cannot have a lock on a at the same time. The region $[2, 3] \times [1, 4]$ represents when both T_1 and T_2 have a lock on b. This region is also forbidden. The set complement of the interior of $[1, 4] \times [2, 3] \cup [2, 3] \times [1, 4]$ in $[0, 5] \times [0, 5]$ is called the Swiss flag and is the Euclidean cubical complex modeling this program design for the dining philosophers problem.

In general, the Euclidean cubical complex modeling a concurrent program is the complement of the interior of the forbidden region. An execution is a directed path from the initial point to the terminal point. Executions are equivalent if they give the same output given the same input, which can be interpreted geometrically as the corresponding paths are dihomotopic in the path space. The Swiss flag has two distinct directed paths up to homotopy equivalence: one corresponding to T_1 using the shared resources first, and the other corresponding to T_2 using the shared resources first. See Fig. 1.

3 Past Links as Obstructions

In this section, we introduce the notions of spaces of directed paths and Euclidean cubical complexes. The (relative) past link of a vertex of a Euclidean cubical complex is defined as a simplicial complex. Studying the contractibility and connectedness of past links gives us insight on the contractibility and connectedness of certain spaces of directed paths.

Definition 1 (d-space) A *d-space* is a pair $(X, \overrightarrow{P}(X))$, where X is a topological space and $\overrightarrow{P}(X) \subseteq P(X) := X^{[0,1]}$ is a family of paths on X (called *dipaths*) that is closed under non-decreasing reparametrizations and concatenations, and contains all constant paths.

For every x, y in X, let $\overrightarrow{P}_x^y(X)$ be the family of *dipaths from x to y*:

$$\overrightarrow{P}_x^y(X) := \{\alpha \in \overrightarrow{P}(X) : \alpha(0) = x \text{ and } \alpha(1) = y\}.$$

In particular, consider the following directed space: the *directed real line* $\overrightarrow{\mathbb{R}}$ is the directed space constructed from the real line whose family of dipaths $\overrightarrow{P}(\mathbb{R})$ consists of all non-decreasing paths. The *Euclidean space* $\overrightarrow{\mathbb{R}^n}$ is the n-fold product $\overrightarrow{\mathbb{R}} \times \cdots \times \overrightarrow{\mathbb{R}}$ with family of dipaths the n-fold product $\overrightarrow{P}(\mathbb{R}^n) = \overrightarrow{P}(\mathbb{R}) \times \cdots \times \overrightarrow{P}(\mathbb{R})$.

Furthermore, we can solely focus on the family of dipaths in a d-space and endow it with the compact open topology.

Definition 2 (Space of Directed Paths) In a d-space $(X, \overrightarrow{P}(X))$, the *space of directed paths* from x to y is the family $\overrightarrow{P}_x^y(X)$ with the compact open topology.

By topologizing the space of directed paths, we may now use topological reasoning and comparison. Since $\overrightarrow{P}_x^y(X)$ does not have directionality, contractibility and other topological features are defined as in the classical case. Moreover, observe that the set $\overrightarrow{P}_x^y(X)$ might have cardinality of the continuum, but is considered trivial if it is homotopy equivalent to a point.

The d-spaces that we consider in this article are constructed from Euclidean cubical complexes. Let $\mathbf{p} = (p_1, \ldots, p_n), \mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n$. We write $\mathbf{p} \preceq \mathbf{q}$ if and only if $p_i \leq q_i$ for all $i = 1, \ldots, n$. Furthermore, we denote by $\mathbf{q} - \mathbf{p} := (q_1 - p_1, \ldots, q_n - p_n)$ the component-wise difference between $\mathbf{q}$ and $\mathbf{p}$, $|\mathbf{p}| := \sum_{i=1}^n p_i$ is the element-wise sum, or one-norm, of $\mathbf{p}$. Similarly to the one-dimensional case, the interval $[\mathbf{p}, \mathbf{q}]$ is defined as $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{p} \preceq \mathbf{x} \preceq \mathbf{q}\}$.

Definition 3 (Euclidean Cubical Complex) Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$. If $\mathbf{q}, \mathbf{p} \in \mathbb{Z}^n$ and $\mathbf{q} - \mathbf{p} \in \{0, 1\}^n$, then the interval $[\mathbf{p}, \mathbf{q}]$ is an *elementary cube in* $\mathbb{R}^n$ of dimension $|\mathbf{q}-\mathbf{p}|$. A *Euclidean cubical complex* $K \subseteq \mathbb{R}^n$ is the union of elementary cubes.

Remark 1 A Euclidean cubical complex K is a subset of $\mathbb{R}^n$ and it has an associated abstract cubical complex. By a slight abuse of notation, we do not distinguish these.

Every cubical complex K inherits the directed structure from the Euclidean space $\overrightarrow{\mathbb{R}^n}$, described after Definition 1. An elementary cube of dimension d is called a d-cube. The m-skeleton of K, denoted by K_m, is the union of all elementary cubes contained in K that have dimension less than or equal to m. The elements of the zero-skeleton are called the vertices of K. A vertex $\mathbf{w} \in K_0$ is said to be *minimal* (resp., *maximal*) if $\mathbf{w} \preceq \mathbf{v}$ (resp., $\mathbf{w} \succeq \mathbf{v}$) for every vertex $\mathbf{v} \in K_0$.

Following [12], we define the (relative) past link of a vertex of a Euclidean cubical complex as a simplicial complex. Let Δ^{n-1} denote the complete simplicial complex with vertices $\{1, \ldots, n\}$. Simplices of Δ^{n-1} is be identified with elements $\mathbf{j} \in \{0, 1\}^n$. That is, every subset $S \subseteq \{1, \ldots, n\}$ is mapped to the n-tuple with entry 1 in the k-th position if k belongs to S and 0 otherwise. The topological space associated to the simplicial complex Δ^{n-1} is the one given by its geometric realization.

Definition 4 (Past Link) In a Euclidean cubical complex K in $\mathbb{R}^n$, the *past link*, $lk^-_{K,\mathbf{w}}(\mathbf{v})$, of a vertex $\mathbf{v}$, with respect to another vertex $\mathbf{w}$ is the simplicial subcomplex of Δ^{n-1} defined as follows: $\mathbf{j} \in lk^-_{K,\mathbf{w}}(\mathbf{v})$ if and only if $[\mathbf{v} - \mathbf{j}, \mathbf{v}] \subseteq K \cap [\mathbf{w}, \mathbf{v}]$.

Remark 2 While K is a *cubical* complex, the past link of a vertex in K is always a *simplicial* complex.

Remark 3 Often the vertex $\mathbf{w}$ and the complex K are understood. In this case, we denote the past link of $\mathbf{v}$ by $lk^-(\mathbf{v})$.

Remark 4 Other definitions of the (past) link are found in the literature. Unlike Definition 4, (past) links are usually subcomplexes of K. However, the (past) links found in other literature are homeomorphic to the (past) link of Definition 4.

In the following example, we show that a vertex $\mathbf{v}$ can have past links with different homotopy type depending on what the initial vertex $\mathbf{w}$ is. We consider as a Euclidean cubical complex the open top box (Fig. 2) and the past links of the vertex $\mathbf{v} = (1, 1, 1)$, with respect to the vertices $\mathbf{w} = \mathbf{0}$ and $\mathbf{w}' = (0, 0, 1)$.

Example 3 (Open Top Box) Let $L \subset \mathbb{R}^3$ be the Euclidean cubical complex consisting of all of the edges and vertices in the elementary cube $[\mathbf{0}, \mathbf{v}]$ and five of the six two-cubes, omitting the elementary two-cube $[(0, 0, 1), \mathbf{v}]$, i.e., the top of the box. Because the elementary one-cube $[\mathbf{v} - (0, 0, 1), \mathbf{v}] \subseteq L \cap [\mathbf{0}, \mathbf{v}] = L$, $lk^-_{L,\mathbf{0}}(\mathbf{v})$ contains the vertex in Δ^2 corresponding to $\mathbf{j} = (0, 0, 1)$. Similarly, because the elementary two-cube $[\mathbf{v} - (0, 1, 1), \mathbf{v}] \subseteq L$, the past link $lk^-_{L,\mathbf{0}}(\mathbf{v})$ contains the edge in Δ^2 corresponding to $\mathbf{j} = (0, 1, 1)$. However, because the elementary two-cube $[\mathbf{v} - (1, 1, 0), \mathbf{v}]$ is not contained in L, $lk^-_{L,\mathbf{0}}(\mathbf{v})$ does not include the edge corresponding to $\mathbf{j} = (1, 1, 0)$. Instead taking the initial vertex to be $\mathbf{w} = (0, 0, 1)$, we get that $lk^-_{L,\mathbf{w}}(\mathbf{v})$ consists of the two vertices corresponding to $\mathbf{j} = (0, 1, 0)$ and $\mathbf{j}' = (1, 0, 0)$. See Fig. 2.

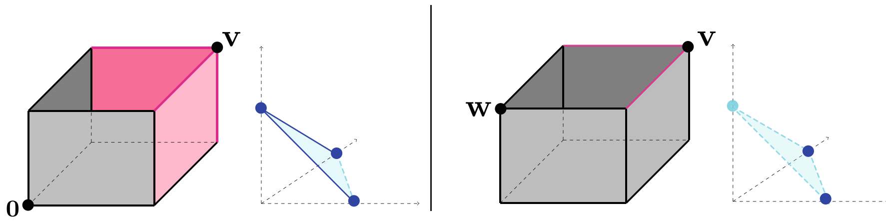


Fig. 2 The Open Top Box. Left: the open top box and the geometric realization of the past link of the red vertex $\mathbf{v} = (1, 1, 1)$, with respect to the black vertex $\mathbf{0}$. The geometric realization of $lk^-_{L,\mathbf{0}}(\mathbf{v})$ contains two edges of a triangle, since the two red faces are included in $[\mathbf{0}, \mathbf{v}]$ and three vertices, since the three red edges are included in $[\mathbf{0}, \mathbf{v}]$. Right: the open top box and the geometric realization of the past link of the red vertex $\mathbf{v} = (1, 1, 1)$, with respect to the black vertex $\mathbf{w} = (0, 0, 1)$. The geometric realization of $lk^-_{L,\mathbf{w}}(\mathbf{v})$ consists only of two vertices of a triangle, since the two red edges are included in $[\mathbf{w}, \mathbf{v}]$

4 The Relationship Between Past Links and Path Spaces

In this section, we illustrate how to use past links to study spaces of directed paths with an initial vertex of $\mathbf{0}$. In particular, the contractibility and connectedness of all past links guarantees the contractibility and connectedness of spaces of directed paths. We also provide a partial converse to the result concerning connectedness.

Theorem 1 (Contractibility) *Let $K \subset \mathbb{R}^n$ be a Euclidean cubical complex with minimal vertex $\mathbf{0}$. Suppose for all $\mathbf{k} \in K_0$, $\mathbf{k} \neq \mathbf{0}$, the past link $lk^-_{\mathbf{0}}(\mathbf{k})$ is contractible. Then, all spaces of directed paths $\vec{P}^{\mathbf{k}}_{\mathbf{0}}(K)$ are contractible.*

Proof By [12, Prop. 5.3], if $\vec{P}^{\mathbf{k}-\mathbf{j}}_{\mathbf{0}}(K)$ is contractible for all $\mathbf{j} \in \{0, 1\}^n$, $\mathbf{j} \neq \mathbf{0}$, and $\mathbf{j} \in lk^-(\mathbf{k})$, then $\vec{P}^{\mathbf{k}}_{\mathbf{0}}(K)$ is homotopy equivalent to $lk^-(\mathbf{k})$. Hence, it suffices to see that all the spaces $\vec{P}^{\mathbf{k}-\mathbf{j}}_{\mathbf{0}}(K)$ are contractible. This follows by structural induction on the partial order on vertices in K.

The start is at $\vec{P}^{\mathbf{0}+\mathbf{e}_i}_{\mathbf{0}}(K)$, where $\mathbf{e}_i$ is the i-th unit vector, and $\mathbf{0} + \mathbf{e}_i \in K_0$. If the edge $[\mathbf{0}, \mathbf{0} + \mathbf{e}_i]$ is in K, then $\vec{P}^{\mathbf{0}+\mathbf{e}_i}_{\mathbf{0}}(K)$ is contractible. Otherwise, $lk^-_{\mathbf{0}}(\mathbf{0} + \mathbf{e}_i)$ is empty, which contradicts the hypothesis that all of the past links are contractible. By structural induction, using also that $\vec{P}^{\mathbf{0}}_{\mathbf{0}}$ is contractible, the theorem now holds. □

Now, we give an analogous sufficient condition for when spaces of directed paths are connected. We provide two different proofs of Theorem 2. The first proof shows how we can use [9, Prop. 2.20] to get our desired result. The second proof uses notions from category theory and is based on the fact that the colimit of connected spaces over a connected category is connected.

Theorem 2 (Connectedness) *With K as above, suppose all past links $lk^-_{\mathbf{0}}(\mathbf{k})$ of all vertices $\mathbf{k} \neq \mathbf{0}$ are connected. Then, for all $\mathbf{k} \in K_0$, all spaces of directed paths $\vec{P}^{\mathbf{k}}_{\mathbf{0}}(K)$ are connected.*

In this first proof we show that [9, Prop. 2.20] is an equivalent condition to all past links being connected.

Proof In [9, Prop. 2.20], a local condition is given that ensures that all spaces of directed paths to a certain final point are connected. Here, we explain how the local condition is equivalent to all past links being connected. Their condition is in terms of the local future; however, we reinterpret this in terms of local past instead of local future. Since we consider all spaces of directed paths *from* a point (as opposed to *to* a point), then reinterpreting the result in terms of local past is the right setting we should look at. The local condition is the following: for each vertex, $\mathbf{v}$, and all pairs of edges $[\mathbf{v}-\mathbf{e}_r, \mathbf{v}]$, $[\mathbf{v}-\mathbf{e}_s, \mathbf{v}]$ in K, there is a sequence of two-cells $\{[\mathbf{v}-\mathbf{e}_{k_i}-\mathbf{e}_{l_i}, \mathbf{v}]\}_{i=1}^{m}$, each of which is in K such that $l_i = k_{i+1}$ for $i = 1, \ldots, m-1$, $k_1 = r$ and $l_m = s$. Now, we show that this local condition is equivalent to ours. In the past link considered as a simplicial complex, such a sequence of two-cells corresponds to a sequence of edges from the vertex r to the vertex s. For $x, y \in lk^-(\mathbf{v})$, they are both connected to a vertex via a line. And those vertices are connected. Hence, the past link is connected.

Vice versa: Suppose $lk^-(\mathbf{v})$ is connected. Let p, q be vertices in $lk^-(\mathbf{v})$ and let $\gamma : I \to lk^-(\mathbf{v}) \in \Delta^{n-1}$ be a path from p to q. The sequence of simplices traversed by γ, $S_1, S_2, \ldots, S_k$, satisfies $S_i \cap S_{i+1} \neq \emptyset$. Moreover, the intersection is a simplex. Let $p_i \in S_i \cap S_{i+1}$. A sequence of pairwise connected edges connecting p to q is constructed by such sequences from p_i to p_{i+1} in S_{i+1} thus providing a sequence of two-cells similar to the requirement in [9]. Hence, by [9], if all past links of all vertices are connected, then all $\vec{P}_{\mathbf{0}}^{\mathbf{k}}$ are connected □

This second proof of Theorem 2 has a more categorical flavor.

Proof We give a more categorical argument which is closer to the proof of Theorem 1. In [10, Prop. 2.3 and Equation 2.2], the space of directed paths $\vec{P}_{\mathbf{0}}^{\mathbf{k}}$ is given as a colimit over $\vec{P}_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}}$. The indexing category is $\mathcal{J}_K$ with objects $\{\mathbf{j} \in \{0,1\}^n : [\mathbf{k}-\mathbf{j}] \subseteq K\}$ and morphisms $\mathbf{j} \to \mathbf{j}'$ for $\mathbf{j} \geq \mathbf{j}'$ given by inclusion of the simplex $\Delta^{\mathbf{j}} \subset \Delta^{\mathbf{j}'}$. The geometric realization of the index category is the past link which with our requirements is connected. The colimit of connected spaces over a connected category is connected. Hence, by induction as above, beginning with edges from $\mathbf{0}$, the directed paths $\vec{P}_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}}$ are all connected and the conclusion follows. □

Remark 5 Our conjecture is that similar results for k-connected past links should follow from the k-connected Nerve Lemma.

Remark 6 The statements of both Theorems 1 and 2 concern past links and path spaces defined with respect to a fixed initial vertex. To see why past links depend on their initial vertex, consider the open top box of Example 3. All past links in L with respect to the initial vertex $\mathbf{0}$ are contractible, but $\vec{P}_{\mathbf{w}'}^{\mathbf{v}}(L)$, where $\mathbf{w}' = (0, 0, 1)$ and $\mathbf{v} = (1, 1, 1)$, is not contractible. It is in fact two points. Note, this does not contradict Theorem 1, which only asserts that $\vec{P}_{\mathbf{0}}^{\mathbf{v}}(L)$ is contractible; see Fig. 2.

We now show how Theorems 1 and 2 can be used to study the spaces of the directed paths in slight modifications of the dining philosophers problem.

Example 4 (Three Concurrent Processes Executing the Same Program) We consider a modification of Example 1 where we have three processes and two resources each with capacity two. All processes are executing the program $P_a P_b V_b V_a$. The Euclidean cubical complex modeling this situation has three dimensions, each representing the program of a process. Since each resource has capacity two, it is not possible to have a three way lock on any of the resources. The three processes have a lock on a in the region $[P_a, V_a]^{\times 3}$, which is the cube $[(1, 1, 1), (4, 4, 4)]$. Similarly, the three processes have a lock on b in the region $[P_b, V_b]^{\times 3}$ which is the cube $[(2, 2, 2), (3, 3, 3)]$. The forbidden region is the union of these two sets which is $[(1, 1, 1), (4, 4, 4)]$. We can model this concurrent program as a three-dimensional Euclidean cubical complex and the forbidden region is the inner $3 \times 3 \times 3$ cube.

In order to analyze the connectedness and contractibility of the spaces of directed paths with initial vertex $\mathbf{0}$, we study the past links of the vertices of K. First, we show that not all past links are contractible. Let $\mathbf{v} = (4, 4, 4)$. Then, $lk^-_{K,\mathbf{0}}(\mathbf{v})$ consists of all $\mathbf{j} \in \{0, 1\}^3$ except $(1, 1, 1)$. The past link does not contain $(1, 1, 1)$ because the cube $[(3, 3, 3), (4, 4, 4)]$ is not contained in K, but $[\mathbf{v} - \mathbf{j}, \mathbf{v}] \subset K$ for all other $\mathbf{j}$. Therefore, $lk^-_{K,\mathbf{0}}(\mathbf{v})$ is the boundary of the two simplex (see Fig. 3). Because the boundary of the two simplex is not contractible, the hypothesis of Theorem 1 is not satisfied. Hence, we cannot use Theorem 1 to study the contractibility of the spaces of directed paths.

Next, we show that all past links are connected. If we directly compute the past link $lk^-_{K,\mathbf{0}}(\mathbf{k})$ for all $\mathbf{k} \in K_0$, we find that the past link consists of either a zero simplex, one simplex, the boundary of the two simplex, or a two simplex. All these past links are connected. Theorem 2 implies that for all $\mathbf{k} \in K_0$, the space of directed paths, $\vec{P}^{\mathbf{k}}_{\mathbf{0}}(K)$ is connected.

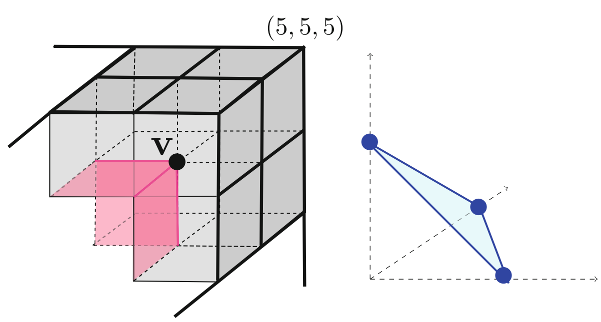


Fig. 3 Three processes, same program. Illustrating $lk^-_{K,\mathbf{0}}(\mathbf{v})$ where K is the cube $[\mathbf{0}, (5, 5, 5)]$ minus the inner cube, $[(1, 1, 1), (4, 4, 4)]$, and $\mathbf{v} = (4, 4, 4)$. The geometric realization of the simplicial complex $lk^-_{K,\mathbf{0}}(\mathbf{v})$ is the boundary of the two simplex since the three pink faces and edges are included in $[\mathbf{0}, \mathbf{v}]$

We can generalize this example to n processes and two resources with capacity $n - 1$ where all processes are executing the program $P_a P_b V_b V_a$. For all n, Theorem 2 shows that all spaces of directed paths are connected.

The converse of Theorem 2 is not true. To see this, and give the conditions under which the converse does hold, we need to introduce the following definition:

Definition 5 (Reachable) The point $x \in K$ is *reachable* from $\mathbf{w} \in K_0$ if there is a path from $\mathbf{w}$ to x. A subcomplex of K is induced by the set of points that are reachable from a vertex $\mathbf{w}$.

Example 5 (Boundary of the $3 \times 3 \times 3$ *Cube with Top Right Cube)* Let K be the Euclidean cubical complex that is the boundary of the $3 \times 3 \times 3$ cube along with the cube $[(2, 2, 2), (3, 3, 3)]$. Observe that all spaces of directed paths with initial vertex $\mathbf{0}$ are connected. However, K has a disconnected past link at $\mathbf{v} = (3, 2, 2)$. If we consider the subcomplex $\hat{K}$ that is reachable from $\mathbf{0}$, then $\hat{K}$ is the boundary of the $3 \times 3 \times 3$ cube. The past links of all vertices in $\hat{K}$ are connected. This motivates the conditions given in Theorem 3 of removing the unreachable points of a Euclidean cubical complex. The connected components of a disconnected past link in the remaining complex can then be represented by directed paths from the initial point and not only locally (Fig. 4).

Theorem 3 (Realizing Obstructions) *Let K be a Euclidean cubical complex with initial vertex* $\mathbf{0}$. *Let $\hat{K} \subset K$ be the subcomplex reachable from* $\mathbf{0}$. *If for $\mathbf{v} \in \hat{K}_0$, the past link in $\hat{K}$ is disconnected, then the path space $\overrightarrow{P}_{\mathbf{0}}^{\mathbf{v}}(K)$ is disconnected.*

Proof Let $\mathbf{v}$ be a vertex such that $lk^-_{K,\mathbf{0}}(\mathbf{v})$ is disconnected and let $\mathbf{j}_1, \mathbf{j}_2$ be vertices in $lk^-_{\hat{K}}(v)$ in different components. The edges $[\mathbf{v} - \mathbf{j}_i, \mathbf{v}]$ are then in $\hat{K}$ and, in particular, $\mathbf{v} - \mathbf{j}_i \in \hat{K}_0$. Hence, there are paths $\mu_i : \overrightarrow{I} \to \hat{K}$ such that $\mu_i(0) = \mathbf{0}$ and $\mu_i(1) = \mathbf{v} - \mathbf{j}_i$.

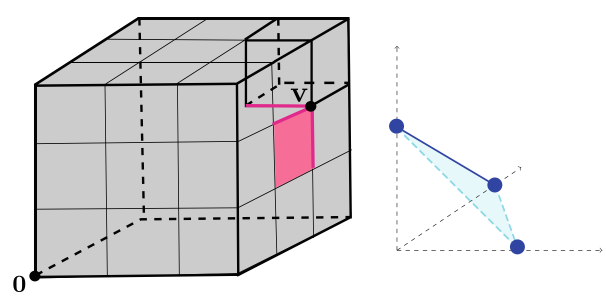


Fig. 4 Motivating reachability condition. Let K be the boundary of the $3 \times 3 \times 3$ cube union with $[(2, 2, 2), (3, 3, 3)]$. Then, the geometric realization of the simplicial complex $lk^-_{K,\mathbf{0}}(\mathbf{v})$ is an edge and a point since the three pink edges and one face are included in $[(0, 0, 0), \mathbf{v}]$

By [3], there are $\hat{\mu}_i$ which are dihomotopic to μ_i and such that $\hat{\mu}_i$ is combinatorial, i.e., a sequence of edges in $\hat{K}$. Let γ_i be the concatenation of $\hat{\mu}_i$ with the edge $[\mathbf{v} - \mathbf{j}_i, \mathbf{v}]$.

Suppose for contradiction that γ_1 and γ_2 are connected by a path in $\vec{P}_{\mathbf{0}}^{\mathbf{v}}(K)$. Let $H : \vec{I} \times I \to K$ be such a path with $H(t, 0) = \gamma_1(t)$ and $H(t, 1) = \gamma_2(t)$. Since $H(t, s)$ is reachable from $\mathbf{0}$, H maps to $\hat{K}$.

By [3], there is a combinatorial approximation $\hat{H} : \vec{I} \times I \to \hat{K}_2$ to the 2-skeleton of $\hat{K} \subset K$. Let B be the open ball centered around $\mathbf{v}$ with radius $1/2$. Since $\hat{H}$ is continuous, the inverse image of B under $\hat{H}$ is a neighborhood of $\{1\} \times I \subset \vec{I} \times I$. For $0 < \epsilon < 1/2$, this neighborhood contains a strip $(1 - \epsilon, 1] \times I$ (by compactness of I). Then $\hat{H}(1 - \epsilon/2 \times I)$ gives a path connecting the two edges $[\mathbf{v} - \mathbf{j}_i, \mathbf{v}]$. This path traverses a sequence of 2-cubes (the carriers). These correspond to a sequence of edges in the past link that connect $\mathbf{j}_1$ and $\mathbf{j}_2$, which contradicts the assumption that they are in different components. Therefore, γ_1 and γ_2 correspond to two points in $\vec{P}_{\mathbf{0}}^{\mathbf{v}}(K)$ that are not connected by a path. □

In general, the reachability condition in Theorem 3 eliminates the spurious disconnected past links that could appear in the unreachable parts of a Euclidean cubical complex.

Example 6 To see how Theorem 3 can be applied, consider Example 2, the Swiss flag. The Swiss flag has two vertices with disconnected past links with respect to $\mathbf{0}$ namely $(4, 3)$ and $(3, 4)$. These disconnected past links imply that Theorem 2 is inconclusive. If the unreachable section of the Swiss flag is removed, we obtain a new Euclidean cubical complex in which the vertex $\mathbf{v} = (4, 4)$ has a disconnected past link, consisting of two points. By Theorem 3, the path space $\vec{P}_{\mathbf{0}}^{\mathbf{v}}(K)$ is also disconnected. In fact, $\vec{P}_{\mathbf{0}}^{\mathbf{v}}(K)$ has two points, representing the dihomotopy classes of paths which pass above the forbidden region, and those paths which pass below.

The disconnected path space, $\vec{P}_{\mathbf{0}}^{\mathbf{v}}(K)$, found in the previous example helps illustrate the following: given two vertices $\mathbf{w}$ and $\mathbf{v}$ in a Euclidean cubical complex K, if the path space $\vec{P}_{\mathbf{w}}^{\mathbf{v}}(K)$ is disconnected, then there exists a vertex in $[\mathbf{w}, \mathbf{v}]$ that has a disconnected past link with respect to $\mathbf{w}$ (the vertices $(4, 3)$ and $(3, 4)$ in the Swiss flag). If $\mathbf{w} = \mathbf{0}$, then we get the contrapositive of Theorem 2. If K is reachable from $\mathbf{0}$, Theorem 3 allows us to draw conclusions about the space of directed paths.

5 Directed Collapsibility

To simplify the underlying topological space of a d-space while preserving topological properties of the associated space of directed paths, we introduce the process of directed collapse. The criteria we require to perform directed collapse on Euclidean cubical complexes involves the topology of the past links of the vertices of the complex. We defined the past links as simplicial complexes that are not themselves directed, so our topological criteria are in the usual sense.

Definition 6 (Directed Collapse) Let K be a Euclidean cubical complex with initial vertex $\mathbf{0}$. Consider $\sigma, \tau \in K$ such that $\tau \subsetneq \sigma$, σ is maximal, and no other maximal cube contains τ. Let $K' = K \setminus \{\gamma \in K | \tau \subseteq \gamma \subseteq \sigma\}$. K' is a *directed (cubical) collapse* of K if, for all $\mathbf{v} \in K_0'$, $lk_K^-(\mathbf{v})$ is homotopy equivalent to $lk_{K'}^-(\mathbf{v})$. The pair τ, σ is then called a *collapsing pair*.

K' is a *directed 0-collapse* of K if for all $\mathbf{v} \in K_0'$, $lk_K^-(\mathbf{v})$ is connected if and only if $lk_{K'}^-(\mathbf{v})$ is connected.

Remark 7 As in the simplicial case, when we remove σ from the abstract cubical complex, the effect on the geometric realization is to remove the interior of the cube corresponding to σ.

Remark 8 Note for finding collapsing pairs, (τ, σ), using Definition 6, with the geometric realization of σ given by the elementary cube, $[\mathbf{w} - \mathbf{j}, \mathbf{w}]$, it is sufficient to only check $\mathbf{v} \in K_0'$ such that $\mathbf{v} = \mathbf{w} - \mathbf{j}'$ where $\mathbf{j} - \mathbf{j}' > 0$. Otherwise the past links, $lk_K^-(\mathbf{v})$ and $lk_{K'}^-(\mathbf{v})$, are equal.

Definition 7 (Past Link Obstruction) Let $\mathbf{w} \in K_0$. A *past link obstruction (type-*∞*)* in K with respect to $\mathbf{w}$ is a vertex $\mathbf{v} \in K_0$ such that $lk_{K,\mathbf{w}}^-(\mathbf{v})$ is not contractible. A *past link obstruction (type-*0*)* in K with respect to $\mathbf{w}$ is a vertex $\mathbf{v} \in K_0$ such that $lk_{K,\mathbf{w}}^-(\mathbf{v})$ is not connected.

Directed collapses preserve some topological properties of the space of directed paths. In particular:

Corollary 1 *If there are no type-*∞ *past link obstructions, then all spaces of directed paths from the initial point are contractible. If there are no type-*0 *past link obstructions, all spaces of directed paths from the initial point are connected.*

Proof Contractibility is a direct consequence of Theorem 1. Likewise, connectedness follows from Theorem 2. □

Corollary 2 (Invariants of Directed Collapse) *If we have a sequence of directed collapses from K to K', then there are no obstructions in K iff there are no obstructions in K'.*

Remark 9 (Past Link Obstructions are Inherently Local) The past link of a vertex is constructed using local (rather than global) information from the cubical complex. Therefore, a past link obstruction is also a local property, which is not dependent on the global construction of the cubical complex.

Below, we provide a few motivating examples for our definition of directed collapse. In general, we want our directed collapses to preserve all spaces of directed paths between the initial vertex and any other vertex in our cubical complex. Notice, τ from Definition 6 is a *free face* of K. Performing a directed collapse with an arbitrary free face of a directed space K with minimal element $\mathbf{0} \in K_0$ and maximal element $\mathbf{1} \in K_0$ can modify the individual spaces of directed paths $\overrightarrow{P}{}_{\mathbf{0}}^{\mathbf{v}}(K)$ and $\overrightarrow{P}{}_{\mathbf{v}}^{\mathbf{1}}(K)$ for $\mathbf{v} \in K_0$.

When $\overrightarrow{P}{}_{\mathbf{v}}^{\mathbf{1}}(K) = \emptyset$, we call $\mathbf{v}$ a *deadlock*. When $\overrightarrow{P}{}_{\mathbf{0}}^{\mathbf{v}}(K) = \emptyset$, we call $\mathbf{v}$ *unreachable*. Deadlocks and unreachable vertices are in a sense each others opposites. Notice if we take the same directed space K yet reverse the direction of all dipaths, then deadlocks become unreachable vertices and vice versa. However, as Examples 7 and 8 illustrate, the creation of an unreachable vertex in the process of a directed collapse might result in a past link obstruction at a neighboring vertex while the creation of a deadlock does not.

Example 7 (3×3 Grid, Deadlocks and Unreachability) Let K be the Euclidean cubical complex in $\mathbb{R}^2$ that is the 3×3 grid. Consider the Euclidean cubical complexes K' and K'' obtained by removing (τ, σ) with $\tau = [(1, 3), (2, 3)]$, $\sigma = [(1, 2), (2, 3)]$ and (τ', σ') with $\tau' = [(1, 0), (2, 0)]$, $\sigma' = [(1, 0), (2, 1)]$, respectively. While K' is a directed collapse of K, K'' is not a directed collapse of K because K'' introduces a past link obstruction at $(2, 1)$. So, (τ, σ) is a collapsing pair while (τ', σ') is not. Collapsing K to K' creates a deadlock at $(1, 3)$ but this does not change the space of directed paths from the designated start vertex $\mathbf{0}$ to any of the vertices between $\mathbf{0}$ and the designated end vertex $(3, 3)$ (see K' in Fig. 5). However, collapsing K to K'' creates an unreachable vertex $(2, 0)$ from the start vertex $\mathbf{0}$ (see K'' in Fig. 5) which does change the space of directed paths from $\mathbf{0}$ to $(2, 0)$ to be empty. Hence not all spaces of directed paths starting at $\mathbf{0}$ are preserved. This motivates our definition of directed collapse.

Our next example shows how directed collapses can be performed with collapsing pairs (τ, σ) when τ is of codimension one and greater.

Example 8 (3×3 grid, Edge and Vertex Collapses) Consider again the Euclidean cubical complex K from Example 7. If we allow a collapsing pair (τ, σ) with τ of dimension greater than 0, we may introduce deadlocks or unreachable vertices. In particular, collapsing the free edge $\tau = [(1, 3), (2, 3)]$ of the top blue square $\sigma =$

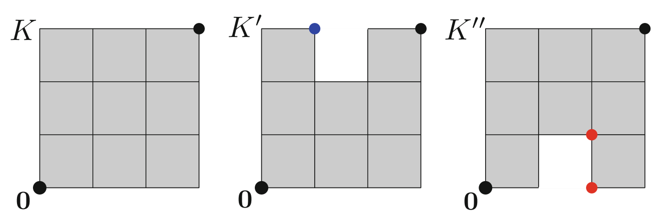
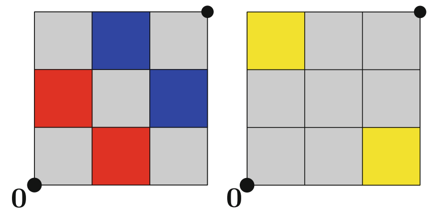


Fig. 5 Illustrating Example 7. On the left: the cubical complex K with initial vertex $\mathbf{0}$ and final vertex $(3, 3)$. In the center: The cubical complex K' which is a directed collapse of K. The deadlock in blue does not change the space of directed paths from $\mathbf{0}$ to any of the vertices between $\mathbf{0}$ and $(3, 3)$. On the right: the cubical complex K'' which is not a directed collapse of K. The space of directed paths into the unreachable red vertex, $(2, 0)$, becomes empty. The empty path space is reflected in the topology of the past link of the red vertex $(2, 1)$ (see Example 8)



(a) Edge Collapse (b) Vertex Collapse

Fig. 6 Illustrating Example 8. On the left: the collapsing of the free edge in the blue squares is an admitted directed collapse. The collapsing of the free edge in the red squares is not an admitted directed collapse. On the right: the collapsing of the free vertex in the yellow squares is an admitted directed collapse

$[(1, 2), (2, 3)]$ in Fig. 6 changes the space of directed paths $\overrightarrow{P}_{(1,3)}^{(3,3)}(K)$ from being trivial to empty in $K\backslash\{\gamma|\tau \subseteq \gamma \subseteq \sigma\}$. Yet we care about preserving the space of directed paths from our designated start vertex $\mathbf{0}$ to any of the vertices (i, j) with $0 \leq i, j \leq 3$ since we ultimately are interested in preserving the path space $\overrightarrow{P}_{\mathbf{0}}^{(3,3)}(K)$. Because of this, such collapses should be allowed in our directed setting. Note that, in these cases, the past link of all vertices remains contractible. However, collapsing the free edge $\tau' = [(1, 0), (2, 0)]$ of the bottom red square $\sigma' = [(1, 0), (2, 1)]$ in Fig. 6 changes the path space $\overrightarrow{P}_{\mathbf{0}}^{(2,0)}(K)$ from being trivial to empty. This change is reflected in the non-contractible past link of $(2, 1)$ in $K\backslash\{\gamma|\tau' \subseteq \gamma \subseteq \sigma'\}$ that consists of the two vertices $\mathbf{j} = (1, 0)$ and $\mathbf{j}' = (0, 1)$ but not the edge $\mathbf{j}'' = (1, 1)$ connecting them. Restricting our collapsing pairs to only include τ of dimension 0 allows for only two potential collapses, the corner vertices $(0, 3)$ and $(3, 0)$ into the yellow squares $[(0, 2), (1, 3)]$ and $[(2, 0), (3, 1)]$, respectively. Neither of these collapses create deadlocks or unreachable vertices and the contractibility of the past link at all vertices is preserved. Performing these corner vertex collapses exposes new free vertices that can be a part of subsequent collapses.

Lastly, we explain how the Swiss flag can be collapsed using a sequence of zero-collapses. The Swiss flag contains uncountably many paths between the initial and final vertex. After performing the sequence of zero-collapses as described in Example 9, the Swiss flag has only two paths up to reparametrization between the initial and final vertex. These two paths represent the two dihomotopy classes of paths that exists for the Swiss flag. Referring back to concurrent programming, we interpret the two paths as two inequivalent executions: either the first process holds a lock on the two resources then releases them so the other process can place a lock on the resources or vice versa.

Example 9 (0-Collapsing the Swiss Flag) The Swiss flag considered as a Euclidean cubical complex in the 5×5 grid has vertices with connected past links, except at $(4, 3)$ and $(3, 4)$. The vertex $(2, 2)$ and the cube $[1, 2] \times [1, 2]$ are a 0-collapsing pair. The vertex $(3, 3)$ and the cube $[3, 4] \times [3, 4]$ are not, since that collapse would produce a disconnected past link at $(4, 4)$. A sequence of 0-collapses preserving

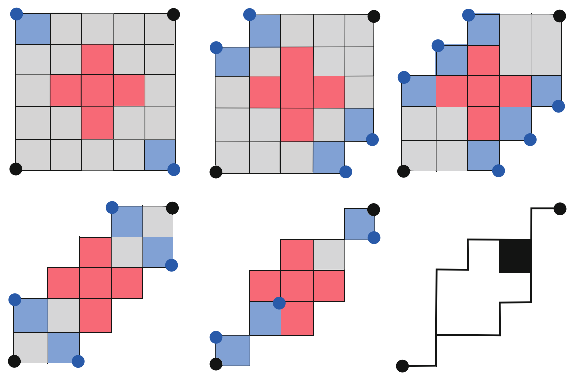


Fig. 7 Zero-collapsing the Swiss Flag. A sequence of zero-collapses is presented from the top left to bottom right. At each stage, the faces and vertices shaded in blue represent the zero-collapsing pairs. The result of the sequence is shown in the bottom right which is a one-dimensional Euclidean cubical complex and one two-cube

the initial and final point will give a one-dimensional Euclidean cubical complex and one 2-cube. Specifically, we get the edges $[0, 1] \times \{0\}$, $\{1\} \times [0, 1]$, $\{1\} \times [1, 3]$, $[1, 3] \times \{1\}$, $[1, 2] \times \{3\}$, $\{3\} \times [1, 2]$, $\{2\} \times [3, 4]$, $[3, 4] \times \{2\}$, $[2, 3] \times \{4\}$, $\{4\} \times [2, 3]$, the square $[3, 4] \times [3, 4]$, and lastly the edges $\{4\} \times [4, 5]$ and $[4, 5] \times \{5\}$ (Fig. 7).

6 Discussion

Directed topological spaces have a rich underlying structure and many interesting applications. The analysis of this structure requires tools that are not fully developed, and a further investigation into these methods will lead to a better understanding of directed spaces. In particular, the development of these notions, such as directed collapse, may lead to a better understanding of equivalence of directed spaces and their spaces of directed paths.

Interestingly, when comparing directed collapse with the notion of cubical collapse in the undirected case, two main contrasts arise. First, the notion of directed collapse is stronger than that of cubical collapse; any directed collapse is a cubical collapse, but not all cubical collapses satisfy the past link requirement of directed collapse. However, directed collapse is not related to existing notions of dihomotopy equivalence which involve continuous maps between topological spaces that preserve directed paths. Hence, directed collapse contrasts cubical collapse in the undirected case since any two spaces related by cubical collapses are homotopic. This contrast suggests the need for dihomotopy equivalence with respect to an initial point.

Directed collapse may not preserve dihomotopy equivalence, so we can collapse more than, e.g., Kahl. By Theorem 2, if K' is a directed collapse of K with respect to $\mathbf{v}$ and K' has trivial spaces of directed paths from $\mathbf{v}$, then so does K. Similarly, if all spaces of directed paths are connected in K', then all spaces of directed paths are connected in K. Hence, our definition of directed collapsibility preserves spaces of directed paths with an initial vertex of $\mathbf{0}$. Preserving spaces of directed paths allows us to study more types of concurrent programs and preserve notions of partial executions.

We plan to pursure many future avenues of research in the directed topological setting. First, we hope to find necessary and sufficient conditions for a pair of cubical cells (τ, σ) to be a collapsing pair. The key will be to have a better understanding of what removing a cubical cell does to the past link of a complex. Additionally, we would like to find directed conterparts to the various types of simplicial collapses. For example, is there a notion of strong directed collapse? As strong collapse also considers the link of a vertex, a consideration of how strong collapse extends to a directed setting seems natural.

Next, we would like to learn more about past link obstructions. We know that performing a directed collapse will not alter the space of directed paths of a Euclidean cubical complex; however, if we are unable to perform a directed collapse due to a past link obstruction, what happens to the space of directed paths? Theorem 3 is a start in understanding what happens to spaces of directed paths for 0 collapses. Another question may be, in what way are obstructions of type ∞ realized as non-contractible spaces of directed paths?

Another direction of research we hope to pursue is defining a way to compute a directed homology that is collapsing invariant. Even the two-dimensional setting (where the cubes are at most dimension two) has proved to be difficult, as adding one two-cell can have various effects, depending on the past links of the vertices involved. We would like to classify the spaces where such a dynamic programming approach would work.

Lastly, many computational questions arise on how to implement the collapse of a directed cubical complex. In [8], an example of collapsing a three-dimensional cubical complex is implemented in C++. This algorithm could be used as a model when handling the directed complex.

Many interesting theoretical and computational questions continue to emerge in the field of directed topology. We hope that our research excites others in studying cubical complexes in the directed setting.

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References

1. Jonathan Ariel Barmak and Elias Gabriel Minian. Strong homotopy types, nerves and collapses. *Discrete Comput. Geom.*, 47(2):301–328, 2012.
2. Edsger W. Dijkstra. Two starvation-free solutions of a general exclusion problem. 1977. Manuscript EWD625, from the archives of UT Austin, https://www.cs.utexas.edu/users/EWD/.
3. Lisbeth Fajstrup. Dipaths and dihomotopies in a cubical complex. *Advances in Applied Mathematics*, 35(2):188–206, 2005.
4. Lisbeth Fajstrup, Eric Goubault, Emmanuel Haucourt, Samuel Mimram, and Martin Raussen. *Directed Algebraic Topology and Concurrency*. Springer Publishing Company, Inc., 1st edition, 2016.
5. Lisbeth Fajstrup, Eric Goubault, and Martin Raussen. Algebraic topology and concurrency. *Theoretical Comuter Science*, pages 241–271, 2006.
6. Robin Forman. A user's guide to discrete Morse theory. *Sém. Lothar. Combin.*, 48, 12 2001.
7. Charles A.R. Hoare. Communicating sequential processes. *Commun. ACM*, 21(8):666–677, August 1978.
8. Jacques-Olivier Lachaud. Cubical complex collapse, 2017.
9. Martin Raussen. On the classification of dipaths in geometric models for concurrency. *Mathematical Structures in Computer Science*, 10(4):427–457, 2000.
10. Martin Raussen and Krzysztof Ziemiański. Homology of spaces of directed paths on euclidean cubical complexes. *Journal of Homotopy and Related Structures*, 9(1):67–84, Apr 2014.
11. John H.C. Whitehead. Simplicial spaces, nuclei and m-groups. *Proceedings of the London Mathematical Society*, s2-45(1):243–327.
12. Krzysztof Ziemiański. On execution spaces of PV-programs. *Theoretical Computer Science*, 619:87–98, 2016.